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ON THE MOSTOW RIGIDITY THEOREM AND MEASURABLE FOLIATIONS BY HYPERBOLIC SPACE†

BY
ROBERT J. ZIMMER††

ABSTRACT
An ergodic-theoretic version of Mostow's rigidity theorem for hyperbolic space forms is obtained treating foliations of a measure space by leaves that carry the structure of a hyperbolic space.

1. Introduction

Suppose G and G' are Lie groups acting in a measure class preserving fashion on measure spaces (S,μ) and (S',μ') respectively. There are then a variety of natural notions of "isomorphism" of the actions. If $G = G'$ one of course has the natural notion of conjugacy of the actions, namely the existence of a bijective (modulo null sets) measure class preserving Borel G-map between S and S'. For arbitrary G and G', one has the notion of orbit equivalence of the actions, that is, the existence of a bijective (modulo null sets) measure class preserving Borel map from S to S' that takes orbits onto orbits. In other words, one asks for isomorphism of the associated measurable equivalence relations defined by the actions. For $G = G'$ this is, of course, a priori a much weaker notion than conjugacy of the actions. There are in addition a variety of intermediate notions of isomorphism that derive from the observation that each orbit of the action inherits a variety of structures from the group and hence one can ask for a bijective (modulo null sets) measure class preserving Borel map from S to S' that takes orbits onto orbits in such a way as to preserve a given structure on the orbits. Thus, for a Lie group acting (essentially) freely, (almost) every orbit will have a topological, differentiable, Riemannian, conformal, etc., structure and so

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†† Sloan Foundation Fellow.
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we can speak of homeomorphic, smooth, isometric, quasi-isometric, conformal, quasi-conformal, etc., orbit equivalences. Providing each orbit with its G-space structure of course just leads to the notion of conjugacy of the actions.

Abstracting the situation somewhat one can more generally speak of a measure space with an equivalence relation such that each equivalence class (or "leaf") has a topological, smooth, etc., structure which vary measurably over the entire measure space. Thus, for example, by a measurable foliation we mean the situation in which every equivalence class has a C^∞-structure and with a suitable local triviality condition holding [15], and by a stratified measure space we mean the situation in which each equivalence class has a locally compact topology [15]. Similarly, we speak of a Riemannian measurable foliation or measurable foliation with conformal structure. Each free ergodic action of a Lie group defines a measurable foliation [15] and smooth orbit equivalence is just isomorphism (or "diffeomorphism") of the associated measurable foliations. In the same manner, homeomorphic, isometric, etc., orbit equivalence is just isomorphism of certain natural associated structures. In certain situations, it is natural to consider not only the measurable foliations defined by the orbits of the Lie group action but also the measurable foliation defined by factoring by the action of a maximal compact subgroup. For example, if G is a connected semisimple Lie group with finite center, $K \subset G$ a maximal compact subgroup, and S is a free ergodic G-space, then S/K has a natural equivalence relation on it in which each leaf has the structure of a Riemannian symmetric space isometric to G/K. (We discuss this more carefully in section 2 below.)

In a recent paper [16] we proved an analogue of the Mostow–Margulis rigidity theorems for ergodic actions of connected semisimple Lie groups with finite center, no compact factors, and of **R**-rank at least 2. This theorem asserts that for irreducible ergodic actions of such groups with finite invariant measure, orbit equivalence of the actions implies first that the groups acting are locally isomorphic, and second that in the centerfree case the actions are conjugate modulo an automorphism of the group. In terms of the associated measurable foliations by symmetric spaces this theorem roughly asserts that isomorphism of the ergodic equivalence relations defined on transversals implies that the Riemannian measurable foliations are isometric modulo normalizing scalar multiples (independent of the leaves). Thus, at least for this class of measurable foliations by symmetric spaces, a purely measure theoretic invariant determines the symmetric Riemannian structure in almost every leaf. These results are all in very sharp contrast to those for actions of amenable groups, and one has for example the result that any two properly ergodic free actions of continuous unimodular amenable groups with finite invariant measure are orbit equivalent [1], [2], [11].

A question we raised in [16] was whether or not the results of that paper extend to the case of simple Lie groups of **R**-rank 1 thus, in particular, including within the above context a class of measurable foliations by hyperbolic space. The point of this paper is to present a result in this direction. Namely, our main theorem is the following.

THEOREM 1. *Let $G = \mathrm{SO}(1,n)/\{\pm I\}$, $n \geqq 3$. Let S, S' be essentially free ergodic G-spaces with finite invariant measure. Let $K \subset G$ be a maximal compact subgroup and S/K and S'/K the associated Riemannian measurable foliations by n-dimensional hyperbolic space G/K. Suppose there is a quasi-conformal homeomorphic orbit equivalence $S/K \to S'/K$. Then the actions of G on S and S' are conjugate modulo an automorphism of G and the associated foliations by hyperbolic space are isometric modulo a normalizing scalar multiple.*

In particular, if the associated Riemannian measurable foliations by hyperbolic space are quasi-isometric (i.e., there is a smooth orbit equivalence with the norm of the derivative and its inverse bounded on almost all leaves) then the actions are conjugate modulo an automorphism of the group. This is again in sharp contrast to the situation for $\mathbf{R}^n$ actions where very different actions can define quasi-isometric Riemannian measurable foliations. (Cf. [6].)

In [16], the assumption that the **R**-rank be at least 2 was necessary in order to use at a basic point a technique used by Margulis in proving arithmeticity of lattices [7]. In proving Theorem 1 we adopt a viewpoint closer to that of Mostow in his proof of rigidity of hyperbolic space forms [9], [10]. In particular, we apply some fundamental results of Mostow on quasi-conformal mappings in higher dimensions which were basic to his original rigidity proof in [9].

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2. Preliminaries

We begin by establishing some basic definitions and recalling some results we will need. Let G be a locally compact second countable group, (S, μ) a standard Borel space with probability measure and suppose G acts on S so that the action map $S \times G \to S$ is Borel and so that μ is quasi-invariant under the action, i.e., for $g \in G$ and $A \subset S$ measurable, $\mu(A) = 0$ if and only if $\mu(Ag) = 0$. We shall often be making the stronger assumption that μ is invariant, i.e., $\mu(Ag) =$

$\mu(A)$. The action is called ergodic if $Ag = A$ for all $g \in G$ and A measurable implies A is null or conull. We call the action free if almost every stabilizer is trivial. If G is a Lie group, then S has the structure of a measurable foliation in the sense of [15], i.e., a measure space with equivalence relation such that each equivalence class has the structure of a C^∞ manifold in such a way that a suitable local triviality condition holds.

Suppose now that G is a connected semisimple Lie group with finite center and that K is a maximal compact subgroup. Define $a : S \times G \to S$ by $a(s, g) = sg$. Let G act on $S \times G$ by $(s, h) \cdot g = (s, hg)$, so that a is then a G-map. If we restrict the G-actions on $S \times G$ and S to K, the quotient spaces will be standard Borel spaces and hence we get an induced map $\tilde{a} : S \times K \backslash G \to S/K$ where we denote the set of cosets gK by $K \backslash G$. The space S/K has an equivalence relation given by the equivalence classes (or leaves) $(s \cdot G)/K$. For each $s \in S$, the map $\lambda_s : K \backslash G \to S/K$ defined by $\lambda_s([g]) = \tilde{a}(s, [g])$ is a Borel map onto $(s \cdot G)/K$ and will be a bijection if the stabilizer of s is trivial. Furthermore, for $h \in G$ we have the equation

$$\lambda_{sh} \circ \beta(h) = \lambda_s, \tag{1}$$

where $\beta(h) : K \backslash G \to K \backslash G$ is left translation by h^{-1}. Thus each leaf in S/K inherits a C^∞-structure from $K \backslash G$ via λ_s and (1) implies that this structure is independent of the choice of s in the leaf. Thus S/K is a smoothly stratified measure space in the sense of [15]. Furthermore, one can show that the local triviality condition in [15] holds and hence S/K is a measurable foliation. (As we shall not make serious use of the local triviality condition, we now only indicate briefly how to obtain it, omitting some measure theoretic details. Let NAK be an Iwasawa decomposition of G and N_0, A_0 compact neighborhoods of the identity in N and A respectively. Then choose a transversal, or complete lacunary section in the sense of [5], T, for the G action on S with respect to the compact set N_0A_0K. Then the image of $T \cdot N_0A_0K$ in S/K will have the required product structure. Choosing a suitable collection of T will produce the required locally trivial covering of S/K.) Now endow $K \backslash G$ with the (unique up to normalizing scalar multiples) G-invariant Riemannian metric converting $K \backslash G$ into a symmetric space. Equation (1) implies that this can unambiguously be transferred to each $(s \cdot G)/K$, and hence S/K is a Riemannian measurable foliation with almost every leaf isometric to $K \backslash G$.

We now recall the notion of a cocycle of an ergodic action. If S is an ergodic G-space and M is a standard Borel group a Borel function $\alpha : S \times G \to M$ is called a cocycle if for all $g, h \in G$, $\alpha(s, gh) = \alpha(s, g)\alpha(sg, h)$ for almost all s. Two cocycles, α, β are called equivalent if there is a Borel function $\varphi : S \to M$ such that for all g, $\varphi(s)\alpha(s,g)\varphi(sg)^{-1} = \beta(s,g)$ a.e. If $\pi : G \to M$ is a homomorphism then $\alpha(s,g) = \pi(g)$ defines a cocycle which we call the restriction of π to $S \times G$. An orbit equivalence also defines a cocycle. More precisely, if S, S' are free ergodic G, G'-spaces, respectively, an orbit equivalence is a measure class preserving Borel bijection $\theta : S_0 \to S'_0$ between conull sets such that orbits are mapped onto orbits. We can then define $\alpha : S \times G \to G'$ (almost everywhere) by $\theta(s)\cdot\alpha(s,g) = \theta(sg)$. If $G = G'$ then one has the following cohomological condition that orbit equivalent actions be conjugate or automorphically conjugate (i.e., conjugate modulo an automorphism of G).

PROPOSITION 2 [16, proposition 2.4]. *Suppose α is a cocyle defined by an orbit equivalence of two free ergodic G-actions, say on S, S'. Then if α is equivalent to the restriction to $S \times G$ of an automorphism of G then the G-actions are automorphically conjugate. If the automorphism is inner, the actions are conjugate.*

Finally, we recall some facts concerning quasi-conformal mappings. For details, the reader is referred to Mostow's paper [9]. Suppose $\varphi : M \to M'$ is a homeomorphism of Riemannian manifolds. For $x \in M$ and $\varepsilon > 0$, let $B(x,\varepsilon)$ be the ε-ball centered at x, $R(x,\varepsilon)$ the radius of the circumscribed ball of $\varphi(B(x,\varepsilon))$ centered at $\varphi(x)$, $r(x,\varepsilon)$ the radius of the inscribed ball of $\varphi(B(x,\varepsilon))$ centered at $\varphi(x)$, and $Q(x) = \limsup_{\varepsilon\to 0} R(x,\varepsilon)/r(x,\varepsilon)$. Then φ is called K-quasi-conformal if $Q(x)$ is bounded on M and $Q(x) \leqq K$ a.e. [9, p. 90], and quasi-conformal if it is K-quasi-conformal for some K. The proofs of the following results can be found in Mostow's paper [9].

THEOREM 3 (Mostow). (a) *A quasi-conformal map is differentiable almost everywhere* [9, theorem 9.1].

(b) *Let $\varphi : M \to M'$ be a quasi-conformal map of Riemannian manifolds and assume φ is differentiable at $p \in M$. Then $Q(p)^2 = \Lambda/\lambda$, where Λ and λ are the maximum and minimum eigenvalues, respectively, of $(d\varphi)^*_p(d\varphi)_p$* [9, p. 89].

(c) *A 1-quasi-conformal mapping $S^n \to S^n$ is conformal if $n \geqq 2$* [9, lemma 12.2].

(d) *Let φ be a quasi-conformal mapping of an open ball in $\mathbf{R}^n$ onto itself. Then φ extends to a homeomorphism of the closed ball and the boundary homeomorphism of the sphere is quasi-conformal* [9, theorems 10.1, 10.2].

3. Proof of Theorem 1

The idea of the proof is the following. On each leaf, we have a quasi-conformal mapping and by Mostow's results these extend to quasi-conformal mappings on the boundary spheres. The formalism we use to deal with this is to pull everything back to $S \times K \backslash G$ rather than work on the foliation itself. We first observe, using Proposition 2, that it suffices to prove that the boundary maps are actually conformal, and by Mostow's results it suffices to show that the boundary maps are 1-quasi-conformal. Our approach to this is to use ergodicity of the restriction of the G-action on S to a minimal parabolic subgroup, and then to show that quasi-conformality without 1-quasi-conformality would yield finite dimensional subrepresentations of the representation of the minimal parabolic on $L^2(S)$. This is impossible by results of C. C. Moore [8]. We now proceed to the proof.

We have the map $a : S \times G \to S$ defining the action and we let $a_s : G \to s \cdot G \subset S$ be given by $a_s(g) = a(s, g)$. We then have the equation $a_{sh} \circ \beta(h) = a_s$ where $\beta(h) : G \to G$ is left translation by h^{-1}. We shall also denote by $\beta(h)$ the map $K \backslash G \to K \backslash G$ given by left translation by h^{-1}. We also recall that we have the maps $\lambda_s : K \backslash G \to (s \cdot G)/K$ as in the preceding section. We shall similarly denote by a', a'_s, etc., the corresponding maps defined for the G-action on S'. Let $\tilde{\theta} : S/K \to S'/K$ be a quasi-conformal equivalence between the Riemannian measurable foliations. Since the G-actions are free and K is compact, as Borel K-spaces we have $S \cong S/K \times K$, $S' = S'/K \times K$ where the K-action is given by right translation on the second factor. Let $\theta : S \to S'$ be defined by $\theta([s], k) = (\tilde{\theta}([s]), k)$. Then θ is a Borel orbit equivalence between the G-actions, and we let $\alpha : S \times G \to G$ be the corresponding cocycle as described in section 2. By Proposition 2, it suffices to show that α is equivalent to the restriction of an automorphism of G. For each $s \in S$, define $\psi_s : G \to G$ by $\psi_s = (a'_{\theta(s)})^{-1} \circ \theta \circ a_s$ where we view $(a'_{\theta(s)})^{-1}$ as a map $\theta(s) \cdot G \to G$. We now claim the following holds:

(2) For almost all s, $\beta(\alpha(s, g)) = \psi_{sg} \circ \beta(g) \circ \psi_s^{-1}$.

This follows by observing that

$$\begin{aligned}
\psi_{sg} &= (a'_{\theta(sg)})^{-1} \circ \theta \circ a_{sg} \\
&= (a'_{\theta(s) \cdot \alpha(s,g)})^{-1} \circ \theta \circ a_s \circ \beta(g)^{-1} \\
&= (a'_{\theta(s)} \circ \beta(\alpha(s, g))^{-1})^{-1} \circ \theta \circ a_s \circ \beta(g)^{-1} \\
&= \beta(\alpha(s, g)) \circ \psi_s \circ \beta(g)^{-1}.
\end{aligned}$$

The maps ψ_s also have the property that they factor to maps $\varphi_s : K \backslash G \to K \backslash G$. To see this, observe that if $x, y \in G$ with $xk = y$ for some $k \in K$, then $\theta(sy) = \theta(sxk) = \theta(sx) \cdot k$, and so

$$\begin{aligned}\theta(s)\cdot\psi_s(y) &= (\theta\circ a_s)(y) \qquad \text{(by definition of } \psi_s\text{)}\\ &= [(\theta\circ a_s)(x)]\cdot k\\ &= \theta(s)\cdot\psi_s(x)k.\end{aligned}$$

By freeness of the action, this implies $\psi_s(y)=\psi_s(x)k$.

Thus from (2) we also have the relation

$$\beta(\alpha(s,g))=\varphi_{sg}\circ\beta(g)\circ\varphi_s^{-1}, \tag{3}$$

where now β is left translation on $K\backslash G$. The maps φ_s can alternatively be given by the formula $\varphi_s=(\lambda'_{\theta(s)})^{-1}\circ\tilde{\theta}\circ\lambda_s$ where λ and λ' are maps as above. The maps λ_s and $\lambda'_{\theta(s)}$ are isometries by definition of the Riemannian structure on the leaves of the measurable foliations and $\tilde{\theta}$ is quasi-conformal. It follows that for almost all s, $\varphi_s: K\backslash G\to K\backslash G$ is a quasi-conformal mapping.

The Riemannian manifold $K\backslash G$ can be identified with the unit ball in $\mathbf{R}^n$ endowed with the Poincare metric and G then acts by isometries. Each isometry extends uniquely to a homeomorphism of the closed ball and one thus has an action of G on the boundary sphere S^{n-1}. As shown in [9], G acts conformally and transitively on S^{n-1}. The stabilizer of a point $p_0\in S^{n-1}$ is a minimal parabolic subgroup $P\subset G$ and we identify the sphere S^{n-1} with $P\backslash G$. Since φ_s is quasi-conformal, by Theorem 3, φ_s extends to a homeomorphism of the closed ball, and we let $\bar{\varphi}_s$ denote the boundary homeomorphism, which is also quasi-conformal. Letting β denote the action of G on $P\backslash G$, from (3) we readily obtain

$$\beta(\alpha(s,g))=\bar{\varphi}_{sg}\circ\beta(g)\circ\bar{\varphi}_s^{-1}. \tag{4}$$

We now claim that to prove the theorem it suffices to show that $\bar{\varphi}_s$ is actually 1-quasi-conformal for almost all s. For then, by Theorem 3, $\bar{\varphi}_s$ is conformal. The conformal group of the sphere can be identified with $O(1,n)/\{\pm I\}$ [9], and hence for each s, we then have an element $\varphi(s)\in O(1,n)/\{\pm I\}$ such that

$$\beta(\alpha(s,g))=\beta(\varphi(sg))\beta(g)\beta(\varphi(s))^{-1}$$

where β represents the action of $O(1,n)/\{\pm I\}$ on S^{n-1}. Since the action of $O(1,n)/\{\pm I\}$ on S^{n-1} is effective this implies that

$$\alpha(s,g)=\varphi(s)^{-1}g\varphi(sg),$$

i.e., α is equivalent as a cocycle into $O(1,n)/\{\pm I\}$ to the restriction of the embedding $G\to O(1,n)/\{\pm I\}$. However, the argument of [13, lemma 3.4] or [12, theorem 6.1] then shows that as cocycle into G, α is equivalent to the restriction of an automorphism of G (which in fact comes from an inner automorphism of $O(1,n)/\{\pm I\}$). The theorem then follows from Proposition 2.

To show that $\bar{\varphi}_s$ is 1-quasi-conformal for almost all s, for $(s,x) \in S \times G/P$ let $C_{(s,x)}$ be the maximal eigenvalue of $d(\bar{\varphi}_s)^*_x d(\bar{\varphi}_s)_x$. Since $\bar{\varphi}_s$ is quasi-conformal for almost all s, $C_{(s,x)}$ is well-defined for almost all (s,x). One can also check that $C_{(s,x)}$ is measurable. Let $V_{(s,x)} \subset T(G/P)_x$ be the subspace consisting of eigenvectors with eigenvalue $C_{(s,x)}$. We can also characterize $V_{(s,x)}$ as $\{v \in T(G/P)_x \mid \| d(\bar{\varphi}_s)_x(v)\|^2 = C_{(s,x)}\| v \|^2\}$. Since G acts conformally on S^{n-1}, by rewriting (4) as $\bar{\varphi}_{sg} = \beta(\alpha(s,g)) \circ \bar{\varphi}_s \circ \beta(g)^{-1}$, it follows that

$$d\beta(g)_x(V_{(s,x)}) = V_{(s,x)\cdot g}. \tag{5}$$

We also note that G acts ergodically on $S \times G/P$ as this is equivalent to the ergodicity of P on S [14, theorem 4.2] which in turn follows by results of C. C. Moore [8]. Equation (5) implies that $\dim(V_{(s,x)})$ is essentially invariant, and so by ergodicity $\dim V_{(s,x)} = k$ for almost all (s,x) where $1 \leqq k \leqq n-1$. To show that φ_s is 1-quasi-conformal for almost all s, it suffices by Theorem 3(b) to show that $k = n-1$. By Fubini, there exists $x \in G/P$ such that $\dim V_{(s,x)} = k$ for almost all $s \in S$ and equation (5) holds for almost all $s \in S$. Let P' be the stabilizer of x in G. Then by restricting to $S \times \{x\}$ we can view V as a map $V: S \to G(n-1,k)$ where the latter is the Grassmann manifold of k-planes in the $n-1$ dimensional space $T(G/P)_x$. The group P' acts on $T(G/P)_x$ via $v \cdot h = d\beta(h)_x(v)$ and equation (5) says that V is essentially a P'-map. By Moore's ergodicity theorem [8], P' is ergodic on S and hence if we let μ be the given measure on S, then $V_*(\mu)$ is a probability measure on $G(n-1,k)$ which is invariant and ergodic under P'. However, it is easily seen that the linear transformations of $T(G/P)_x$ defined by elements of P' form exactly the group of similarities of this inner product space, and hence each P'-orbit in $G(n-1,k)$ is closed. This implies that each P'-ergodic measure on $G(n-1,k)$ is supported on an orbit [3]. Thus we can view V as a P'-map $V: S \to P'/P_0$ where $P_0 \subset P'$ is the stabilizer of an element in $G(n-1,k)$. We can write $P' = MAN$ where M is isomorphic to the $n-2$ dimensional special orthogonal group and acts on S^{n-1} leaving x fixed, and A and N are the group of conformal transformations of S^{n-1} corresponding under stereographic projection of $S^{n-1} - \{x\}$ onto the space tangent to the antipodal of x, to the positive scalar multiples and the translations in the tangent space, respectively. (See [9, p. 98] for example.) Furthermore, N acts trivially on $T(G/P)_x$ [10, 20.18], A acts by scalar multiplications, and AN is normal in P'. Therefore, $AN \subset P_0$ and the P' action on P'/P_0 factors to a $P'/AN \cong M$ action. In particular, by the compactness of M, $L^2(P'/P_0)$ is a direct sum of finite dimensional P'-invariant subspaces, where the action of P' on $L^2(P'/P_0)$ is translation. However, the P'-map $S \to P'/P_0$ shows that $L^2(P'/P_0) \subset L^2(S)$ corresponds to a subrepresentation of the representation of P' on $L^2(S)$. By Moore's theorem [8], this implies $L^2(P'/P_0)$ is the constants, i.e., that $P_0 = P'$, so that P' leaves an element in $G(n-1,k)$ invariant. This clearly implies $k = n-1$, and as remarked above, this suffices to prove Theorem 1.

4. Concluding remarks

(a) Theorem 1 is of course also true for actions of $\mathrm{O}(1,n)/\{\pm I\}$ and the proof in fact shows that for these groups the conclusion can be strengthened to assert that the actions are actually conjugate (rather than conjugate modulo an automorphism).

(b) The proof of Theorem 1 also shows the following.

THEOREM 4. *Let G, K, S, S' be as in Theorem 1. Suppose the measurable foliations S/K and S'/K are smoothly isomorphic in such a way that almost all the diffeomorphisms between leaves extend to diffeomorphisms of the boundary spheres. Then the G-actions on S and S' are conjugate modulo an automorphism of G coming from an inner automorphism of* $\mathrm{O}(1,n)/\{\pm I\}$.

(c) If $(X,\mathcal{F})$ is any ergodic Riemannian measurable foliation by hyperbolic n-space, then one can show that $(X,\mathcal{F})$ is derived, via the above construction, from an ergodic action of $\mathrm{O}(1,n)/\{\pm 1\}$. If $(X,\mathcal{F})$ is any ergodic measurable foliation and T is a transversal, then there is an assigned measure class on T that is quasi-invariant under the partial automorphisms of the equivalence relation on T defined by intersection with the leaves. We say that the foliation has a holonomy invariant measure if there is a measure in this measure class which is actually invariant under the partial automorphisms. (This is independent of the transversal.) If there is a Riemannian structure on each leaf, then there is a measure on X that locally is the integral, with respect to the invariant measure on the transversals, of the measures on the leaf determined by the Riemannian structure. We say that the foliation has finite total volume if this measure is finite on X. These remarks and Theorem 1 for $\mathrm{O}(1,n)/\{\pm 1\}$ actions enable us to deduce the following.

THEOREM 5. *Let $(X_i,\mathcal{F}_i)$, $i=1,2$, be ergodic measurable foliations by hyperbolic n-space, $n \geqq 3$, and assume both foliations have a holonomy invariant measure and finite total volume. If the foliations are quasi-conformally equivalent, then they are isometric modulo a normalizing scalar multiple (independent of the leaf).*

(d) One has the following questions.

(1) Can Theorem 1 be extended to actions of other **R**-rank 1 simple Lie groups, perhaps using ideas of [10, sections 19–23]?

(2) Can a free ergodic action of $SO(1, n)$ with finite invariant measure be orbit equivalent to such an action of $SO(1, k)$? More generally, do the results of [16] extend to the **R**-rank 1 case?

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHICAGO
CHICAGO, IL 60637 USA